## Section A: Pure Mathematics

**1** Show that

$$\int_{\pi/6}^{\pi/4} \frac{1}{1-\cos 2\theta}\ \mathrm{d}\theta = \frac{\sqrt{3}}{2} - \frac{1}{2}\ .$$

By using the substitution $x = \sin 2\theta$, or otherwise, show that

$$\int_{\sqrt{3}/2}^{1} \frac{1}{1-\sqrt{(1-x^2)}}\ \mathrm{d}x = \sqrt{3} - 1 - \frac{\pi}{6}\ .$$

Hence evaluate the integral

$$\int_{1}^{2/\sqrt{3}} \frac{1}{y(y-\sqrt{(y^2-1^2)})}\ \mathrm{d}y\ .$$

**2** Show that setting $z - z^{-1} = w$ in the quartic equation

$$z^4 + 5z^3 + 4z^2 - 5z + 1 = 0$$

results in the quadratic equation $w^2 + 5w + 6 = 0$. Hence solve the above quartic equation.

Solve similarly the equation

$$2z^8 - 3z^7 - 12z^6 + 12z^5 + 22z^4 - 12z^3 - 12z^2 + 3z + 2 = 0\ .$$

**3** The $n$th Fermat number, $F_n$, is defined by

$$F_n = 2^{2^n} + 1\,, \qquad n = 0,\ 1,\ 2,\ \ldots\ ,$$

where $2^{2^n}$ means 2 raised to the power $2^n$. Calculate $F_0$, $F_1$, $F_2$ and $F_3$. Show that, for $k = 1$, $k = 2$ and $k = 3$,

$$F_0 F_1 \ldots F_{k-1} = F_k - 2\ . \tag{$*$}$$

Prove, by induction, or otherwise, that $(*)$ holds for all $k \geqslant 1$. Deduce that no two Fermat numbers have a common factor greater than 1.

Hence show that there are infinitely many prime numbers.

**4** Give a sketch to show that, if $\mathrm{f}(x) > 0$ for $p < x < q$, then $\int_p^q \mathrm{f}(x)\,\mathrm{d}x > 0$.

(i) By considering $\mathrm{f}(x) = ax^2 - bx + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $3b < 2a + 6c$.

(ii) By considering $\mathrm{f}(x) = a\sin^2 x - b\sin x + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $4b < (a + 2c)\pi$.

(iii) Show that, if $a > 0$, $b^2 < 4ac$ and $q > p > 0$, then

$$b\ln(q/p) < a\left(\frac{1}{p} - \frac{1}{q}\right) + c(q - p)\ .$$

**5** The numbers $x_n$, where $n = 0, 1, 2, \ldots$, satisfy

$$x_{n+1} = kx_n(1 - x_n)\ .$$

(i) Prove that, if $0 < k < 4$ and $0 < x_0 < 1$, then $0 < x_n < 1$ for all $n$.

(ii) Given that $x_0 = x_1 = x_2 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, find $k$ in terms of $a$.

(iii) Given instead that $x_0 = x_2 = x_4 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, show that $ab^3 - b^2 + (1 - a) = 0$, where $b = k(1 - a)$. Given, in addition, that $x_1 \neq a$, find the possible values of $k$ in terms of $a$.

**6** The lines $l_1$, $l_2$ and $l_3$ lie in an inclined plane $P$ and pass through a common point $A$. The line $l_2$ is a line of greatest slope in $P$. The line $l_1$ is perpendicular to $l_3$ and makes an acute angle $\alpha$ with $l_2$. The angles between the horizontal and $l_1$, $l_2$ and $l_3$ are $\pi/6$, $\beta$ and $\pi/4$, respectively. Show that $\cos\alpha\sin\beta = \frac{1}{2}$ and find the value of $\sin\alpha\sin\beta$. Deduce that $\beta = \pi/3$.

The lines $l_1$ and $l_3$ are rotated in $P$ about $A$ so that $l_1$ and $l_3$ remain perpendicular to each other. The new acute angle between $l_1$ and $l_2$ is $\theta$. The new angles which $l_1$ and $l_3$ make with the horizontal are $\phi$ and $2\phi$, respectively. Show that

$$\tan^2\theta = \frac{3 + \sqrt{13}}{2}\ .$$

**7** In 3-dimensional space, the lines $m_1$ and $m_2$ pass through the origin and have directions $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$, respectively. Find the directions of the two lines $m_3$ and $m_4$ that pass through the origin and make angles of $\pi/4$ with both $m_1$ and $m_2$. Find also the cosine of the acute angle between $m_3$ and $m_4$.

The points $A$ and $B$ lie on $m_1$ and $m_2$ respectively, and are each at distance $\lambda\sqrt{2}$ units from $O$. The points $P$ and $Q$ lie on $m_3$ and $m_4$ respectively, and are each at distance 1 unit from $O$. If all the coordinates (with respect to axes $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) of $A$, $B$, $P$ and $Q$ are non-negative, prove that:

(i) there are only two values of $\lambda$ for which $AQ$ is perpendicular to $BP$;

(ii) there are no non-zero values of $\lambda$ for which $AQ$ and $BP$ intersect.

**8** Find $y$ in terms of $x$, given that:

$$\text{for } x<0\,,\quad \frac{\mathrm{d}y}{\mathrm{d}x}=-y \quad\text{and}\quad y=a \text{ when } x=-1\,;$$
$$\text{for } x>0\,,\quad \frac{\mathrm{d}y}{\mathrm{d}x}=y \quad\text{and}\quad y=b \text{ when } x=1\,.$$

Sketch a solution curve. Determine the condition on $a$ and $b$ for the solution curve to be continuous (that is, for there to be no 'jump' in the value of $y$) at $x=0$.

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=|\,\mathrm{e}^x-1|\,y$$

given that $y=\mathrm{e}^{\mathrm{e}}$ when $x=1$ and that $y$ is continuous at $x=0$. Write down the following limits:

$$\text{(i)}\ \lim_{x\to+\infty} y\exp(-\mathrm{e}^x)\,; \qquad \text{(ii)}\ \lim_{x\to-\infty} y\,\mathrm{e}^{-x}\,.$$

## Section B: Mechanics

**9** A particle is projected from a point $O$ on a horizontal plane with speed $V$ and at an angle of elevation $\alpha$. The vertical plane in which the motion takes place is perpendicular to two vertical walls, both of height $h$, at distances $a$ and $b$ from $O$. Given that the particle just passes over the walls, find $\tan\alpha$ in terms of $a$, $b$ and $h$ and show that

$$\frac{2V^2}{g} = \frac{ab}{h} + \frac{(a+b)^2 h}{ab} \, .$$

The heights of the walls are now increased by the same small positive amount $\delta h$. A second particle is projected so that it just passes over both walls, and the new angle and speed of projection are $\alpha + \delta\alpha$ and $V + \delta V$, respectively. Show that

$$\sec^2\alpha \, \delta\alpha \approx \frac{a+b}{ab} \, \delta h \, ,$$

and deduce that $\delta\alpha > 0$. Show also that $\delta V$ is positive if $h > ab/(a+b)$ and negative if $h < ab/(a+b)$.

**10** A competitor in a Marathon of $42\frac{3}{8}$ km runs the first $t$ hours of the race at a constant speed of 13 km h$^{-1}$ and the remainder at a constant speed of $14 + 2t/T$ km h$^{-1}$, where $T$ hours is her time for the race. Show that the minimum possible value of $T$ over all possible values of $t$ is 3.

The speed of another competitor decreases linearly with respect to time from 16 km h$^{-1}$ at the start of the race. If both of these competitors have a run time of 3 hours, find the maximum distance between them at any stage of the race.

**11** A rigid straight beam $AB$ has length $l$ and weight $W$. Its weight per unit length at a distance $x$ from $B$ is $\alpha W l^{-1}(x/l)^{\alpha-1}$, where $\alpha$ is a positive constant. Show that the centre of mass of the beam is at a distance $\alpha l/(\alpha+1)$ from $B$.

The beam is placed with the end $A$ on a rough horizontal floor and the end $B$ resting against a rough vertical wall. The beam is in a vertical plane at right angles to the plane of the wall and makes an angle of $\theta$ with the floor. The coefficient of friction between the floor and the beam is $\mu$ and the coefficient of friction between the wall and the beam is also $\mu$. Show that, if the equilibrium is limiting at both $A$ and $B$, then

$$\tan\theta = \frac{1-\alpha\mu^2}{(1+\alpha)\mu} \, .$$

Given that $\alpha = 3/2$ and given also that the beam slides for any $\theta < \pi/4$ find the greatest possible value of $\mu$.

## Section C: Probability and Statistics

**12** On $K$ consecutive days each of $L$ identical coins is thrown $M$ times. For each coin, the probability of throwing a head in any one throw is $p$ (where $0 < p < 1$). Show that the probability that on exactly $k$ of these days more than $l$ of the coins will each produce fewer than $m$ heads can be approximated by

$$\binom{K}{k} q^k (1-q)^{K-k},$$

where

$$q = \Phi\left(\frac{2h - 2l - 1}{2\sqrt{h}}\right), \qquad h = L\Phi\left(\frac{2m - 1 - 2Mp}{2\sqrt{Mp(1-p)}}\right)$$

and $\Phi(.)$ is the cumulative distribution function of a standard normal variate.

Would you expect this approximation to be accurate in the case $K = 7$, $k = 2$, $L = 500$, $l = 4$, $M = 100$, $m = 48$ and $p = 0.6$ ?

**13** Let $\mathrm{F}(x)$ be the cumulative distribution function of a random variable $X$, which satisfies $\mathrm{F}(a) = 0$ and $\mathrm{F}(b) = 1$, where $a > 0$. Let

$$\mathrm{G}(y) = \frac{\mathrm{F}(y)}{2 - \mathrm{F}(y)} \ .$$

Show that $\mathrm{G}(a) = 0$, $\mathrm{G}(b) = 1$ and that $\mathrm{G}'(y) \geqslant 0$. Show also that

$$\frac{1}{2} \leqslant \frac{2}{(2 - \mathrm{F}(y))^2} \leqslant 2 \ .$$

The random variable $Y$ has cumulative distribution function $\mathrm{G}(y)$. Show that

$$\tfrac{1}{2}\,\mathrm{E}(X) \leqslant \mathrm{E}(Y) \leqslant 2\mathrm{E}(X) \ ,$$

and that

$$\mathrm{Var}\,(Y) \leqslant 2\,\mathrm{Var}\,(X) + \tfrac{7}{4}(\mathrm{E}(X))^2 \ .$$

**14** A densely populated circular island is divided into $N$ concentric regions $R_1$, $R_2$, ..., $R_N$, such that the inner and outer radii of $R_n$ are $n-1$ km and $n$ km, respectively. The average number of road accidents that occur in any one day in $R_n$ is $2 - n/N$, independently of the number of accidents in any other region.

Each day an observer selects a region at random, with a probability that is proportional to the area of the region, and records the number of road accidents, $X$, that occur in it. Show that, in the long term, the average number of recorded accidents per day will be

$$2 - \frac{1}{6}\left(1 + \frac{1}{N}\right)\left(4 - \frac{1}{N}\right) .$$

[Note: $\sum\limits_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$ .]

Show also that

$$\mathrm{P}(X = k) = \frac{\mathrm{e}^{-2}N^{-k-2}}{k!}\sum_{n=1}^{N}(2n-1)(2N-n)^k\,\mathrm{e}^{n/N} .$$

Suppose now that $N = 3$ and that, on a particular day, two accidents were recorded. Show that the probability that $R_2$ had been selected is

$$\frac{48}{48 + 45\,\mathrm{e}^{1/3} + 25\,\mathrm{e}^{-1/3}} .$$